THE FINITENESS PROBLEM FOR AUTOMATON SEMIGROUPS IS UNDECIDABLE

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Abstract. The finiteness problem for automaton groups and semigroups has been widely studied, several partial positive results are known. However we prove that, in the most general case, the problem is undecidable.

We study the case of automaton semigroups. Given a NW-deterministic Wang tile set, we construct a Mealy automaton, such that the plane admit a valid Wang tiling if and only if the Mealy automaton generates a finite semigroup. The construction is similar to a construction by Kari for proving that the nilpotency problem for cellular automata is unsolvable.

Moreover Kari proves that the tiling of the plane is undecidable for NW-deterministic Wang tile set. It follows that the finiteness problem for automaton semigroup is undecidable.

1. Introduction

Automaton groups, where first introduced by Gluškov in [5]. This family of group is a powerful tool to build example or counter-example to various problems in group theory. Alešin in [2] constructs a new counter-example to the unbounded Burnside problem.

Grigorčuk gave in [6] an infinite 2-group G generated by three involutions, giving an other counter-example to the unbounded Burnside problem. Grigorčuk solves the Milnor problem in [8, 7], proving that G is of intermediate growth (its growth is neither polynomial nor exponential).

Wilson in [16] answer a question by Gromov, constructing an example of group with exponential growth but whitout uniform exponential growth.

An automaton group is generated by the state of a finite Mealy automaton. Therefore it is natural to ask which classical group-theoretical question are decidable.

For example, the word problem is decidable. There is an algorithm which, given an automaton group (or automaton semigroup) and words in the generators, decide wether the words represent the same elements.

On the other hands Šunić and Ventura construct in [14] examples of automaton groups in which the conjugacy problem is not solvable.

We refere to [9, Section 7], for a list of several decision problem on automaton semigroup. The finiteness problem has been widely stuied, several partial positive results are known. Klimann proves in [12] that the finiteness problem is solvable among invertible-reversible Mealy automata with two states (or two letters).

In this paper we prove that the finiteness problem for automaton semigroup is not solvable. Note that several partial positive results are known (cf. [12, 1]).

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The proof relies on a construction by Kari in [10]. Kari construct, given a NW-deterministic tile set T, a celular automata C_T, such that the plane has valid tiling in T if and only if C_T is not nilpotent. Kari also prove that the tiling problem for NW-deterministic tile set is unsolvable, hence the nilpotency problem for cellular automata is undecidable.

Cellular automaton are similar to Mealy automaon, Kari's construction of in [10] can be adapted to Mealy automata. Given a NW-deterministic tile set T we construct a Mealy automaton $\mathcal{A}_T$ such that the plane has valid tiling in T if and only if $\langle \mathcal{A}_T \rangle_+$ is infinite, hence the finiteness problem for automata group is undecidable.

The problem is still open for automaton groups. Although the methods of Lecerf in [13], the result of Kari and Ollinger in [11], proving that periodicity is undecidable for cellular automata, suggest that the finiteness problem is also undecidable for automaton group.

2. Basic concepts

Given a set X, we denote by X^n the set of all words of length n over X, that is the set of all sequences $(x_1, \dots, x_n)$ in X. We denote by X^ω the set of all infinite sequences $(x_k)_{k \in \mathbb{N}}$ in X. We denote X^* the set of all finite words. Given $u \in X^*$ and $v \in X^* \cup X^\omega$, we denote by uv the concatenation of the words u and v. We also denote

$$X^* = \bigcup_{n<\omega} X^n$$

$$X^{\leq n} = \{u \in X^* \mid \operatorname{lh} u \leq n\} = \bigcup_{k \leq n} X^k$$

$$X^{<n} = \{u \in X^* \mid \operatorname{lh} u < n\} = \bigcup_{k<n} X^k$$

Given $x \in X$ we denote by x^n the constant sequence of length n which take the value x, and by $x^\omega = (x)_{k \in \mathbb{N}}$ the infinite constant sequence.

A Mealy automaton $\mathcal{A}$ is a 4-tupple $(A, \Sigma, \delta, \sigma)$ where A and Σ are finite sets, $\delta \colon A \times \Sigma \to A$ and $\sigma \colon A \times \Sigma \to \Sigma$ are maps.

We extend the maps $\sigma \colon A^* \times \Sigma^{\leq\omega} \to \Sigma^{\leq\omega}$ and $\delta \colon A^{\leq\omega} \times \Sigma^* \to A^{\leq\omega}$ in the usual way. We also denote $\sigma_a(u) = \sigma(a, u)$ and $\delta_u(a) = \delta(a, u)$, for all $a \in A^*$ and all $u \in \Sigma^*$. The equalities (2.1)-(2.4) are satisfied, indeed these equalities define the extensions of the maps δ and σ.

$$\sigma_a(uv) = \sigma_a(u)\sigma_{\delta_u(a)}(v)\,, \quad \text{for all } a \in A^*,\ u \in \Sigma^*, \text{ and } v \in \Sigma^* \cup \Sigma^\omega. \tag{2.1}$$

$$\delta_u(ab) = \delta_u(a)\delta_{\sigma_a(u)}(b)\,, \quad \text{for all } u \in \Sigma^*,\ a \in A^*, \text{ and } b \in A^* \cup A^\omega. \tag{2.2}$$

$$\sigma_{ab} = \sigma_b \circ \sigma_a\,, \quad \text{for all } a, b \in A^*. \tag{2.3}$$

$$\delta_{uv} = \delta_v \circ \delta_u\,, \quad \text{for all } u, v \in \Sigma^*. \tag{2.4}$$

Note that, given $a \in A^*$, the map σ_a preserves the length of each word $u \in \Sigma^{\leq\omega}$, moreover if u is a prefix of v, then $\sigma_a(u)$ is a prefix of $\sigma_a(v)$. That is σ_a is an endomorphism of the tree σ^*.

We denote by $\langle \mathcal{A} \rangle_+$ the subsemigroup of $\operatorname{End} \Sigma^*$ generated by $\{\sigma_a \mid a \in A\}$, equivalentely $\langle \mathcal{A} \rangle_+ = \{\sigma_a \mid a \in A^*\}$.

3. Main

The following definition is due to Wang [15].

Definition 3.1. A *Wang tile* is a tuple $t = (t_N, t_S, t_E, t_W)$, viewed as a square with colored edges. A *tile set* is a finite set of Wang tile. A *Wang tiling* of a subset S of $\mathbb{Z}^2$, with a tile set T, is a map $t\colon S \to T$. We say that t is *valid* if, given $(x, y) \in \mathbb{Z}^2$, the following equalities hold

$$\begin{aligned} t(x,y)_N &= t(x,y+1)_S, && \text{if } (x,y) \in S \text{ and } (x,y+1) \in S.\\ t(x,y)_E &= t(x+1,y)_W, && \text{if } (x,y) \in S \text{ and } (x+1,y) \in S. \end{aligned}$$

A simple compactness argument give the following classical result.

Theorem 3.2. *Let T be a tile set. The set $\mathbb{Z}^2$ has a valid Wang tiling if and only if each finite subset of $\mathbb{Z}^2$ has a valid Wang tiling.*

Remark 3.3. In particular, if $\mathbb{Z}^2$ has no valid Wang tiling, then there is a least integer $n \in \mathbb{N}$ such that $\{0, 1, \dots, n\}^2$ has no valid Wang tiling.

The existence of valid Wang tiling is hard to determined, as illustrate the following result of R. Berger in [3].

Theorem 3.4 (Berger)**.** *It is undecidable wether a finite tile set has a valid Wang tiling for $\mathbb{Z}^2$.*

The following notion was introduced by Kari in [10].

Definition 3.5. A tile set T is *NW-deterministic* if each tile is determined by the north and west colors. That is $t_N = s_N$ and $t_W = s_W$ imply that $t = s$, for all $s, t \in T$.

Theorem 3.4 is generalized by Kari in [10].

Theorem 3.6 (Kari)**.** *It is undecidable wether a finite NW-deterministic tile set has a valid Wang tiling for $\mathbb{Z}^2$.*

The main goal was to generalize a result of Culik, Pachl, and Yu in [4] to dimension one. Kari prove the following theorem in [10].

Theorem 3.7 (Kari)**.** *It is undecidable wether a one-dimensional cellular automaton is nilpotent.*

The argument can be adapted to automaton semigroups, however we need to be careful about a side effect (Cellular automata act on words indexed by $\mathbb{Z}$, each element of an automaton semigroup acts on words indexed by $\mathbb{N}$. We first define a Mealy automaton from a tile set (Kari uses a similar construction to obtain a cellular automata).

Definition 3.8. Let T be a finite NW-deterministic tile set. The *Mealy automata of T* is the tuple $\mathcal{A}_T = (Q, \Sigma, \delta, \sigma)$, where $Q = \Sigma = T \sqcup \{\perp\}$, and the maps δ and σ are defined by

$$\begin{aligned} \delta\colon Q \times \Sigma &\to Q\\ (x, y) &\mapsto y \end{aligned}$$

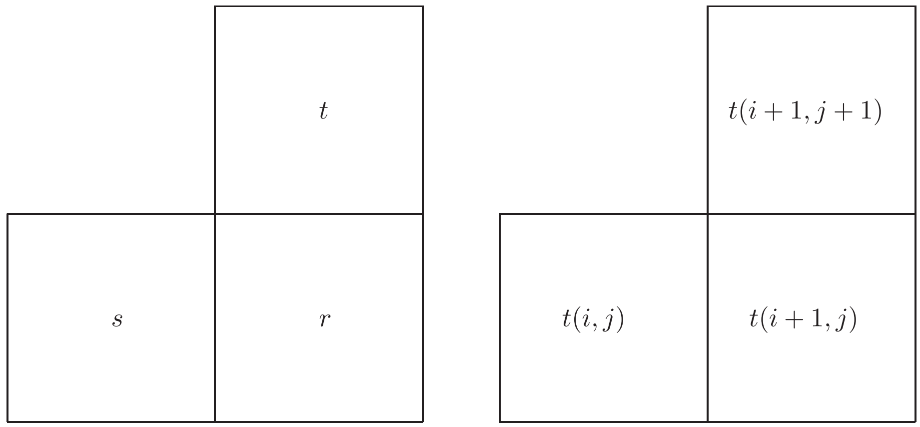

FIGURE 1. Wang tilings.

The new state does not depend on the old one, the automaton only remember the previous letter.

$$\begin{aligned}
\sigma\colon Q \times \Sigma &\to \Sigma \\
(\perp, s) &\mapsto \perp \\
(t, \perp) &\mapsto \perp \\
(\perp, \perp) &\mapsto \perp \\
(s, t) &\mapsto r && \text{if } r_N = t_S \text{ and } r_W = s_E. \\
(s, t) &\mapsto \perp && \text{otherwise.}
\end{aligned}$$

That is, given $s, t, r \in T$, if the Wang tiling on the left hand side of Figure 1 is valid, then $\sigma(s, t) = r$, in all other cases $\sigma(s, t) = \perp$.

Remark 3.9. The Mealy automaon of a finite NW-deterministic tile set T should be understood in the following way. A word w in Q, can be seen as tiles on the diagonal of a square, the Mealy automaton transform this word, to the corresponding tiles that can be placed bellow the diagonal. If it is imposible to put a tile at some place, then the "mistake" tile $\perp$ is placed instead.

Remark 3.10. Note that $\delta_x(y) = y$ for all $x \in Q$ and $y \in \Sigma$. It follows that:

$$\sigma_x(u) = \sigma_x(u_0)(\sigma_{u_k}(u_{k+1}))_{k\in\mathbb{N}}, \quad \text{for all } u = (u_k)_{k\in\mathbb{N}} \in \Sigma^\omega, \text{ and all } x \in Q. \tag{3.1}$$

Lemma 3.11. *Let T be a finite NW-deterministic tile set. Let $t\colon \mathbb{Z}^2 \to T$ be a valid Wang tiling. Consider the word $w_n = (t(k+n, k))_{k\in\mathbb{N}}$ for each $n \in \mathbb{N}$. The equality $\sigma_\perp^m(w_n) = \perp^m w_{m+n}$ holds for all $n, m \in \mathbb{N}$. In particular all the maps $\sigma_\perp^m$ are different.*

Proof. We use the notations of Definition 3.8. As the Wang tiling on the right hand side of Figure 1 is valid, it follows that

$$\sigma_{t(i,j)}(t(i+1, j+1)) = t(i+1, j), \quad \text{for all } i, j \in \mathbb{N}. \tag{3.2}$$

Given $n \in \mathbb{N}$, the following equalities hold.

$$\begin{aligned}
\sigma_\perp(w_n) &= \sigma_\perp(t(n,0))(\sigma_{t(n+k,k)}(t(n+k+1,k+1)))_{k\in\mathbb{N}}\,, && \text{by (3.1).}\\
&= \perp(t(n+k+1,k))_{k\in\mathbb{N}}\,, && \text{by (3.2).}\\
&= \perp w_{n+1}
\end{aligned}$$

The result follows by induction. □

Lemma 3.12. *Let T be a finite NW-deterministic tile set. If $\mathbb{Z}^2$ has no valid Wang tiling then $\langle \mathcal{A}_T\rangle_+$ is finite.*

Proof. We use the notations of Definition 3.8. By Theorem 3.2 there is $n \in \mathbb{N}$ such that the set $\{0,1,\dots,n\}^2$ has no valid Wang tiling for T.

Claim. *Let $u \in Q^{2n}$. The following equality holds.*

$$\sigma_u(xy) = \sigma_u(x)\perp^\omega\,, \quad \textit{for all } x \in \Sigma^n \textit{ and all } y \in \Sigma^\omega.$$

Proof of Claim. We can write $u = u_1 \dots u_{2n}$. Set $\tau_0 = \mathrm{id}$, and set:

$$\tau_k = \sigma_{u_1u_2\dots u_k} = \sigma_{u_k}\circ\sigma_{u_{k-1}}\circ\cdots\circ\sigma_{u_1}\,, \quad \text{for each } 1 \le k \le 2n.$$

Notice that

$$\sigma_{u_{k+1}}\circ\tau_k = \tau_{k+1}\,, \quad \text{for all } 0 \le k \le 2n-1. \tag{3.3}$$

Let $x \in \Sigma^n$, let $y \in \Sigma^\omega$. Denote by $f(i,j)$ the $(j+1)^{\text{th}}$ letter of $\tau_i(xy)$, for $(i,j) \in \mathbb{N}^2$ such that $i \le 2n$. That is:

$$\tau_i(xy) = f(i,j)_{j\in\mathbb{N}}\,, \quad \text{for all } 0 \le i \le 2n. \tag{3.4}$$

Given $0 \le i < 2n$, the following equalities hold:

$$\begin{aligned}
f(i+1,j)_{j\in\mathbb{N}} &= \tau_{i+1}(xy)\,, && \text{by (3.4)}\\
&= \sigma_{u_{i+1}}(\tau_i(xy))\,, && \text{by (3.3)}\\
&= \sigma_{u_{i+1}}(f(i,j)_{j\in\mathbb{N}})\,, && \text{by (3.4)}\\
&= \sigma_{u_{i+1}}(f(i,0))(\sigma_{f(i,j)}(f(i,j+1)))_{j\in\mathbb{N}}\,, && \text{by (3.1) in Remark 3.10}
\end{aligned}$$

Therefore the following statement holds

$$\sigma_{f(i,j)}(f(i,j+1)) = f(i+1,j+1)\,, \quad \text{for all } (i,j) \in \mathbb{N}^2 \text{ with } 0 \le i < 2n. \tag{3.5}$$

Assume that $f(2n, n+k) \ne \perp$ for some $k \in \mathbb{N}$. Applying inductively (3.5), with Definition 3.8 we obtain that $f(i+j, i+k) \ne \perp$ for all $0 \le i,j \le n$, and the Wang tiling on Figure 2 is valid.

Therefore $\{0,1,\dots,n\}^2$ has a valid Wang tiling; a contradiction. □ Claim.

Let $u \in Q^*$ be a word of length at least $2n$, let $v \in Q^{2n}$ and $w \in Q$ such that $u = vw$. Let $x \in \Sigma^n$, let $y \in \Sigma^\omega$. We have

$$\sigma_u(xy) = \sigma_{vw}(xy) = \sigma_w(\sigma_v(xy)) = \sigma_w(\sigma_v(x)\perp^\omega) = \sigma_{vw}(x)\perp^\omega = \sigma_u(x)\perp^\omega\,.$$

Therefore $\{\sigma_u \mid u \in Q^* \text{ and } \operatorname{lh} u \ge 2n\}$ is of cardinality at most $\operatorname{card}(\Sigma^n)^{(\Sigma^n)}$.

However $\langle \mathcal{A}_T\rangle_+ = \{\sigma_u \mid u \in Q^{<2n}\} \cup \{\sigma_u \mid u \in Q^* \text{ and } \operatorname{lh} u \ge 2n\}$, therefore the following inequality holds

$$\operatorname{card}\langle \mathcal{A}_T\rangle_+ \le 1 + \operatorname{card} Q + \operatorname{card} Q^2 + \cdots + \operatorname{card} Q^{2n-1} + \operatorname{card}(\Sigma^n)^{(\Sigma^n)}\,.$$

Hence $\langle \mathcal{A}_T\rangle_+$ is finite. □

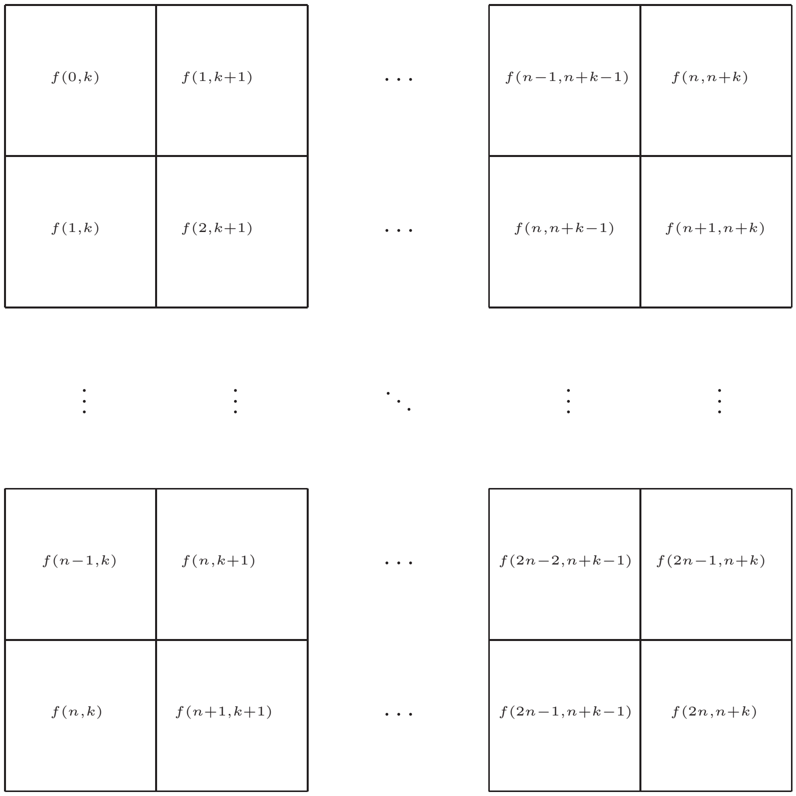

FIGURE 2. A Wang tiling defined by an element of $\langle \mathcal{A}_T \rangle_+$.

From Lemma 3.11 and Lemma 3.12 we see that the existence of a valid Wang tiling of $\mathbb{Z}^2$ is equivalent to the infiniteness of an explicit automaton semigroup. Therefore, from Theorem 3.6 we deduce the following result.

Theorem 3.13. *It is undecidable wether a given automaton semigroup is finite.*

From the proof of Lemma 3.12, we see the following corollary.

Corollary 3.14. *It is undecidable wether, given an automaton semigroup A and $f, c \in A$, there exists n such that $f^n = c$.*

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